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Integrable system constructed out of two interacting superconformal fields

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Abstract. We consider an interaction between superconformal fields of the same gradation. This entails the construction of a supersymmetric Poisson tensor for these fields, generating a new class of Hamiltonian systems. The Lax representation is found for one of them by supersymmetrizing the Lax operator for the Hirota–Satsuma equation. The supersymmetric equation is not reducible to the classical Hirota–Satsuma case. We show that our generalized system can be reduced to the the supersymmetric KdV equation ($a = 4$). Surprisingly the integrals of motion are not reduced to integrals of motion of the supersymmetric KdV equation.

1. Introduction

The Korteweg–de Vries (KdV) equation, which has been extensively studied by mathematicians as well as physicists [1] in the last 30 years, is probably the most popular soliton equation. It bears a deep relation to conformal field theory [2], two-dimensional gravity and matrix models [3].

In this context Gervais [4] discovered that the KdV hierarchies are related, via the second Hamiltonian structure, to Virasoro algebra. This observation has also been extended to other Lie algebras. For example, the nonlinear Schrödinger equation is connected with the $SL(2, C)$ Kac–Moody algebra [5], the Boussinesq equation is connected with the so called W_3 algebra [6].

On the other hand, various different generalizations of the soliton equation have recently been proposed as the Kadomtsev–Petviashvilli and Gelfand–Diki hierarchies and supersymmetrization. The motivation for studying these are diverse. In the supersymmetric generalization, one expects that, in the so called bosonic limit of supersymmetry (SUSY), a new class of the integrable models may appear. Until now supersymmetric KdV hierarchies [7–16] have been constructed for $N = 1$, 2, 3 and 4, based on their relation to superconformal algebras. For an extended $N = 2$ supersymmetric route the Boussinesq [17, 18], nonlinear Schrödinger equation [5, 19–21] and the multicomponent Kodomtsev–Petviashvilli hierarchy [22] have been supersymmetrized as well.

It appears that in order to get a supersymmetric SUSY theory, we have to extend a system of k bosonic equations by kN fermion and $k(N-1)$ boson fields ($k = 1, 2, \ldots, N = 1, 2, \ldots$) in such a way that the final theory becomes SUSY invariant. Interestingly enough, during supersymmetrizations, some typical SUSY effects (compared with classical theory) have occured. We mention a few of them: the nonuniqueness of the roots for the SUSY Lax operator [15], the lack of the bosonic reduction to classical equations (for example in the SUSY Boussinesq equation [17]) and the occurence of nonlocal conservation laws [23].

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In this paper we investigate how it is possible to build the Hamiltonian operator (Poisson tensor) and an integrable system by using two interacting (super)conformal fields. It is possible to carry out such a construction for a (SUSY) Boussinesq equation in which case, we have two conformal fields with different conformal dimensions. However, we are interested in the construction employing two different fields of the same conformal dimension.

Initially, we shall study the classical aspect of our problem, without any reference to SUSY and then we shall consider the supersymmetrizations.

In the 'classical' section we show that it is possible to construct several different Poisson tensors by using two conformal fields of the same dimension. We carry out this construction by assuming that in the limiting case, when the second field vanishes, our Poisson tensor reduces to the tensor that is connected with the Virasoro algebra and hence reproduces the Korteweg–de Vries equation. Among those different Poisson tensors, there is a tensor which could be used to construct the Hirota–Satsuma equation [24]. This equation is a nontrivial extension of the Korteweg–de Vries equation which is integrable, has the Lax operator [25] and the recursion operator [26]. Moreover, in the limiting case of a pure KdV equation (when the second field vanishes), integrability is preserved, and the Lax operator is reduced to the KdV counterpart.

In the supersymmetric case, presented in section 2, we have a much more complicated situation compared with the classical one. First, we carry out a classification of all possible supersymmetric Poisson tensors constructed of two superconformal fields of the same dimensions. For this purpose we use a symbolic computer language, Reduce [27] and the computer package SUSY2 [28]. By analogy with the classical case, we assume these tensors to be reducible to tensors connected with the $N = 2$ super Virasoro algebra and hence to those which reproduce SUSY generalizations of the KdV equation. The SUSY ($N = 2$) extension of the KdV equation leads to a class of equations containing one free parameter, however, only three members of this class ($a = 1, 4, -2$) are integrable and possess Lax pairs. Therefore, again using the computer package SUSY2, we investigate the Lax operator. We assume the most general form on the Lax operator, which reduces to known Lax operators of the SUSY KdV equations. We show that it only reproduces a consistent equation, for a system which is reduced to the SUSY KdV ($a = 4$) equation. Finally we present three nontrivial Hamiltonians for our system.

As a result, in the bosonic limit of our system, we obtain a complicated system of four interacting classical fields. Surprisingly, these equations are not reduced to the classical Hirota–Satsuma equation. This situation seems to be generic because, as mentioned earlier, we encounter the same situation for the super extension of the Boussinesq equation—the lack of the proper bosonic limit. There is also a second aspect of our supersymmetrization: namely, conservations laws of our super system do not coincide, in the limit of the pure SUSY KdV equation, with the conservations laws of SUSY KdV ($a = 4$) equation. We prove that by showing the absence of the second, fourth and sixth conformal dimensional integrals of motions in our generalization. Note that this supersymmetrization of the Hirota–Satsuma equation is integrable due to the existence of the Lax operator.

2. Classical Poisson tensor and Hirota–Satsuma equation

Let us start our consideration with the well known KdV equation

$$u_t = -u_{xxx} + 6uu_x \tag{1}$$

which can be viewed as a Hamiltonin system

$$u_t = \{u, H\} \tag{2}$$

with the Hamiltonian and the Poisson brackets defined by

$$H = \tfrac{1}{2}\int u^2\,\mathrm{d}x \tag{3}$$

$$\{u(x), u(y)\} = (-\partial^3 + 2u\partial + 2\partial u)\delta(x-y). \tag{4}$$

For later use, let us rewrite this equation in an equivalent form by using the Poisson tensor

$$P_2 = -\partial^3 + 2u\partial + 2\partial u \tag{5}$$

$$u_t = P_2\,\mathrm{grad}(H) \tag{6}$$

where grad denotes the functional gradient.

For the Fourier modes of $u(x)$,

$$u(x) = \frac{6}{c}\sum_{n=-\infty}^{\infty}\exp(-\mathrm{i}nx)L_n - \frac{1}{4} \tag{7}$$

the Poisson brackets in equation (4) imply the structure relations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + cn(n^2-1)\delta_{n,m} \tag{8}$$

where c is a central extension term.

It is well known that this equation is completely integrable with an infinite number of integrals of motion in involution. An interesting problem in the theory of solitons is to generalize the KdV equation to a system of equations, in such a way, that the integrability is preserved and in the limiting case, where additional fields vanish, we recover the KdV equation. At the moment, there are many distinct proposals, one of them uses a Poisson tensor constructed of two different conformal fields u and w of the same conformal dimension. Taking into account that the field u is two dimensional, while the usual Poisson tensor of KdV equation is three dimensional, we make the following ansatz

$$P_2 = \begin{pmatrix} c_1\partial_x^3 + z_1k\,\mathrm{d}(u) & c_2\partial_x^3 + z_2k\,\mathrm{d}(u) + z_3k\,\mathrm{d}(w) \\ c_2\partial_x^3 + z_2k\,\mathrm{d}(u) + z_3k\,\mathrm{d}(w) & c_3\partial_x^3 + z_4k\,\mathrm{d}(u) + z_5k\,\mathrm{d}(w) \end{pmatrix} \tag{9}$$

where $c_1, \ldots z_1, \ldots$ are (at the moment) free coefficients and

$$k\,\mathrm{d}(u) = u\partial_x + \partial_x u. \tag{10}$$

In order to obtain the conditions on coefficients c_i and z_i we verify the Jacobi identity [29]

$$\langle a, P'[Pb]c\rangle + \text{cyclic permutation of } (a, b, c) = 0 \tag{11}$$

where a, b, c are arbitrary elements of the real-linear space S [29], while $\langle\rangle$ is a scalar product in S, $P'[Pb]$ denotes the directional derivative in S and is defined as follows

$$P(u)'[Pb] = \frac{\partial}{\partial\epsilon}P(u + \epsilon[Pb])|_{\epsilon=0}. \tag{12}$$

We obtain three different solutions for the coefficients c_i and z_i:

$$z_2 = z_3 = z_4 = c_2 = 0 \tag{13}$$

$$z_3 = 0 \qquad z_5 = \frac{z_2^2 - z_1z_4}{z_2} \qquad c_1 = \frac{c_2z_1}{z_2} \tag{14}$$

$$z_2 = 0 \qquad z_1 = z_3 \qquad c_3 = \frac{c_1z_4 + c_2z_5}{z_3}. \tag{15}$$

The first solution give us the direct product of two standard Virasoro structures (equation (5)) with arbitrary central charges c_1 and c_3. We can apply this Poisson tensor to the gradient

$$H = \tfrac{1}{2}\int uw\,\mathrm{d}x \tag{16}$$

and obtain the equation considered in [30] in the context of the extended supersymmetric ($N = 3$) KdV system.

The second solution is not interesting from our point of view, because it is impossible to reduce this tensor to the standard Virasoro-type Poisson tensor in the usual manner. In order to show that, let us briefly explain the standard Dirac reduction formula [6].

Let U, V be two linear spaces with coordinates u and v. Let

$$P(u, v) = \begin{pmatrix} P_{uu}, & P_{uv} \\ P_{vu}, & P_{vv} \end{pmatrix} \tag{17}$$

be a Poisson tensor on $U \bigoplus V$. Assume that P_{vv} is invertible, then

$$P = P_{uu} - P_{uv}P_{vv}^{-1}P_{vu} \tag{18}$$

is a Poisson tensor on U.

As we see, for the second solution in space where $w = 0$ the reduction is possible if $c_2 = 0$ and $z_2 = 0$, but then, we obtain an undefined central extension term. However, it is interesting to note that we can carry out the reduction in a different way. We can deform this self-consistent structure:

$$w \to z_2 w \qquad \frac{c_2}{z_2} \to k \qquad c_2 \to 0 \qquad z_2 \to 0 \tag{19}$$

and obtain the desired result. On the other hand it is possible to make a reduction in space where $u = 0$ by assuming that $c_2 = 0$ and obtaining the standard Virasoro-type tensor for the field w.

The third solution (15) is the most interesting, since it allows us to make a reduction in the space where $w = 0$ assuming that $c_2 = 0$. This class of Poisson tensors includes the Hamiltonian operator responsible for the Hirota–Satsuma equation, which has the form

$$P_2 = \begin{pmatrix} \partial^3 + \partial u + u\partial & \partial w + w\partial \\ \partial w + w\partial & 2\partial^3 + 2\partial u + 2u\partial \end{pmatrix}. \tag{20}$$

We see that the interaction is concentrated both on the diagonal and off-diagonal elements of the Poisson tensor (19). Therefore we can state that we have constructed an extended Virasoro algebra which contains the usual conformal algebra interacting with an additional conformal field.

The Hamiltonian and equations of motion for the Hirota–Satsuma system are

$$H = \tfrac{1}{2}\int \mathrm{d}x\,(u^2 - w^2) \tag{21}$$

$$u_t = u_{xxx} + 3uu_x - 3ww_x \tag{22}$$

$$w_t = -2w_{xxx} - 3uw_x. \tag{23}$$

Hirota and Satsuma found [24] five nontrivial integrals of motion and later it was proved that this equation is integrable, due to the existence of its Lax representations [25]

$$L = (\partial^2 + u + w)(\partial^2 + u - w) \tag{24}$$

$$L_t = [L, (L^{\frac{3}{4}})_+] \tag{25}$$

where $(+)$ denotes a projection onto the pure differential part of the operator.

3. The extended supersymmetrization of a Poisson tensor constructed of two fields

The basic objects in the supersymmetric analysis are: the superfield and the supersymmetric derivative. We shall deal with the so called extended $N = 2$ supersymmetry for which superfields are superfermions or superbosons. They depend, in addition to x and t, on two anticommuting variables, θ_1 and θ_2, ($\theta_2\theta_1 = -\theta_1\theta_2$, $\theta_1^2 = \theta_2^2 = 0$). Their Taylor expansion with respect to θ is

$$U(x, \theta_1, \theta_2) = u_0(x) + \theta_1\zeta_1(x) + \theta_2\zeta_2(x) + \theta_2\theta_1 u_1(x) \tag{26}$$

where the fields u_0, u_1, are interpreted as the boson (fermion) fields for superboson (superfermion) field and ζ_1, ζ_2, as fermions (bosons) for superboson (superfermion) respectively. The superderivatives are defined as

$$\mathcal{D}_1 = \partial_{\theta_1} + \theta_1\partial \qquad \mathcal{D}_2 = \partial_{\theta_2} + \theta_2\partial \tag{27}$$

with the properties

$$\mathcal{D}_2\mathcal{D}_1 + \mathcal{D}_1\mathcal{D}_2 = 0 \qquad \mathcal{D}_1^2 = \mathcal{D}_2^2 = \partial. \tag{28}$$

Below we shall use the following notation: $(\mathcal{D}_i F)$ denotes the outcome of the action of the superderivative on the superfield, while $\mathcal{D}_i F$ denotes the action itself.

The supersymmetric Poisson tensor connected with Virasora algebra has the form

$$P = cD_1D_2\partial + zs(U) \tag{29}$$

$$s(U) = 2\partial U + 2U\partial - D_1UD_1 - D_2UD_2 \tag{30}$$

where c is the central extension term and z an arbitrary free parameter.We assume that in the SUSY case, the analogue of formula (9) reads

$$P_2 = \begin{pmatrix} c_1D_1D_2\partial + z_1s(U) & c_2D_1D_2\partial + z_2s(U) + z_3s(W) \\ c_2D_1D_2\partial + z_2s(U) + z_3s(W) & c_3D_1D_2\partial + z_4s(U) + z_5s(W) \end{pmatrix}. \tag{31}$$

We checked the Jacobi identity by using the same formula as in the classical case and obtained the same conditions on the central extension terms c_i and z_i. For the same reasons, as in the classical case, we consider the last solution only, assuming additionally that $c_2 = 0$.

It is easy to obtain a Hamiltonian system using the supersymmetric analogue of formula (6). In order to do that we should specify the Hamiltonian. We assume its most general form which has gradation three. Such a Hamiltonian is constructed of all possible combinations of two fields and their (SUSY) derivatives. It is defined by modulo (SUSY) divergent terms and has the following form

$$\begin{aligned} H = \int \mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, (&a_1(D_1D_2U)U + a_2(D_1D_2U)W \\ &+a_3(D_1D_2W)W + a_4W^3 + a_5W^2U + a_6WU^2 + a_7U^3 + a_8W_xU) \end{aligned} \tag{32}$$

where a_i are arbitrary coefficients, a superboson U is defined by equation (25) while a superboson W is

$$W = w_0 + \theta_1\xi_1 + \theta_2\xi_2 + \theta_2\theta_1 w_1 \tag{33}$$

where ξ_i are the fermion-valued functions and w_i are classical functions. Integration over θ in (32) is understood by the Berezin integration

$$\int \theta_j\, \mathrm{d}\theta_1 = \delta_{ij} \qquad \int \theta_i = 0. \tag{34}$$

In this manner it is possible to obtain a huge class of complicated Hamiltonian systems which contain many free parameters. It is easy to extract those systems of equations in which the bosonic limits are reduced to the Hirota–Satsuma equation. Indeed, assuming that

$$z1 = \tfrac{1}{2} \qquad z2 = z5 = c2 = 0 \qquad z3 = z4 = 1 \qquad c1 = -1 \qquad c3 = -2 \tag{35}$$

and choosing

$$H = \int \mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \tfrac{1}{2}((D_1D_2U)U - (D_1D_2W)W + aU^3) \tag{36}$$

where a is an arbitrary constant, we obtain the minimal SUSY generalization of the Hirota–Satsuma equation which also contains the SUSY generalization of the KdV equation

$$\begin{aligned} U_t = \partial[&U_{xx} + (D_1U)(D_2U)(3a + \tfrac{1}{2}) + (D_1D_2U)U(1-3a) \\ &+ \tfrac{3}{2}aU^3 - 2(D_1D_2W)W - (D_2W)(D_1W)] \end{aligned} \tag{37}$$

$$\begin{aligned} W_t = \partial[&-2W_{xx} + 2W(D_1D_2U) - 2U(D_1D_2W) + 3aW^2U] \\ &-(D_2W)(D_1U_x) - (D_2W_x)(D_1U) + (D_1W_x)(D_2U) - (D_1W)(D_2U_x). \end{aligned} \tag{38}$$

4. The strategy and results

We have seen in the previous section that it is possible to obtain a new class of Hamiltonian systems. We would like to find, in this class, an integrable system which contains the Hirota–Satsuma equation in the bosonic limit. Therefore we apply the following strategy in order to solve our problem:

(1) we assume the equations of motion on the superbosons U and W are of the form which is obtained by application of the Poisson tensor (31), with the conditions (15) and $c_2 = 0$, to the gradient of Hamiltonian (32).

(2) We construct the most general SUSY generalization of the 'classical' Lax operator appearing in the Hirota–Satsuma equations (22), (23) and investigate a supersymmetric generalization of its Lax pair (24).

(3) We use the equations of motion constructed in the first approach to verify the validity of the Lax pair obtained in the second approach. In this manner we obtain the system of algebraic equations on the free parameters which appear in the Lax operator, as well as in the equations of motion and in the Poisson tensor. We would like to solve this system of equations.

Before presenting the results of our computations let us briefly recall some basic facts on SUSY $N = 2$ generalizations of the KdV equation which are needed for this construction. This generalization can be written as

$$\begin{aligned} U_t &= P \,\mathrm{grad}\left(\frac{1}{2}U(D_1D_2U) + \frac{a}{3}U^3\right) \\ &= \partial(-U_{xx} + (2+a)U(D_1D_2U) + (a-2)(D_1U)(D_2U) + aU^3) \end{aligned} \tag{39}$$

where P is defined by (29) and a is an arbitrary parameter. It appears that this SUSY generalization is only integrable for three values of parameters a. The integrability can be concluded from the observation that it is possible to find Lax operators [10, 16] for these cases.

The Lax operator in the supersymmetric case is an element of the super pseudo-differential algebra G whose each element g can be represented as:

$$G \ni g = \sum_{n=-\infty}^{\infty} \Phi_n \partial^n = \sum_{n=-\infty}^{\infty} (B_n + F_nD_1 + FF_nD_2 + BB_nD_1D_2)\partial^n \tag{40}$$

where B_i and BB_i are arbitrary superbosons while F_i and FF_i are arbitrary superfermions. In our case of SUSY KdV generalization, the Lax operators are given by:

$$a=-2: \quad L = \partial^2 + D_1 U D_2 - D_2 U D_1 \tag{41}$$

$$\begin{aligned} a=4: \quad L &= \partial^2 - (D_1 D_2 U) - U^2 + (D_2 U) D_1 - (D_1 U) D_2 - 2U D_1 D_2 \\ &= -(D_1 D_2 + U)^2 \end{aligned} \tag{42}$$

$$a=1: \quad L = \partial - \partial^{-1} D_1 D_2 U. \tag{43}$$

For the first two cases we have the usual Lax pair [10]

$$\frac{\partial L}{\partial t} = 4[L, L_+^{\frac{3}{2}}] \tag{44}$$

while for the last case we have the nonstandard Lax pair [16]

$$\frac{\partial L}{\partial t} = [L, L_{\leqslant 1}^3] \tag{45}$$

where $L_{\leqslant 1}^3$ denotes the projection on the subspace

$$P_{\leqslant 1}(\Gamma) = \sum_{n=1}^{\infty} \Phi_n \partial^n + (F_0 D_1 + FF_0 D_2 + BB_0 D_1 D_2). \tag{46}$$

We shall not consider, the nonstandard representation below, as it does not exist for the classical Hirota–Satsuma system.

We have odd and even dimensional integrals of motion for the $a=4$ case. Odd integrals contains the usual conservation laws of the KdV equation, while even integrals do not have such a property. Explicitly, we have the first four integrals of motion for $a=4$

$$I_1 = \int U \, \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \tag{47}$$

$$I_2 = \int U^2 \, \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \tag{48}$$

$$I_3 = \int ((D_1 D_2 U) U + \tfrac{4}{3} U^3) \, \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \tag{49}$$

$$I_4 = \int (U_x^2 + 3(D_1 D_2 U) U^2 + 2U^4) \, \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2. \tag{50}$$

These integrals can be computed by using the following formulae

$$I_{2k+1} = \int \mathrm{Tr}\, L_1^{2k+1} \, \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \tag{51}$$

$$I_{2k} = \int \mathrm{Tr}(L_1 L_2)^k \, \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \tag{52}$$

where Tr denotes trace formula defined on the SUSY pseudo-diferential algebra G. We adopt a definition of Tr such as that which denotes the element standing before $D_1 D_2 \partial^{-1}$ in the algebra G. L_1 and L_2 are two different roots of the Lax operator (equation (42)) where L_1 has the standard form as $\partial + \cdots$, and L_2 is $D_1 D_2 + U$.

In order to construct a Lax operator for our generalization we assumed that it has the following reprsentation

$$L = \partial^4 + \Phi_1 \partial^3 + \Phi_2 \partial^2 + \Phi_3 \partial + \Phi_4 \tag{53}$$

where Φ_i are SUSY operators of ith conformal dimension constructed of all possible combinations of $D_1, D_2, D_1 D_2, U, W,$ (SUSY) derivatives of U, W and with free parameters. It is a huge expression which contains 243 terms (in other words 243 free parameters). We make two additional assumptions: first: in the limit $W = 0$ we should recover Lax operator for the SUSY KdV equation in the form of equation (41) or (42). Second: our ansatz should be O(2) invariant under the change of the supersymmetric derivatives ($D_1 \mapsto -D_2$, $D_2 \mapsto D_1$). This invariance follows from a physical assumption on the nonprivileging of the 'fermionic' coordinates in the superspace.

These assumptions simplify our ansatz on the Lax operator giving 208 terms for the $a = 4$ case, but only 195 terms for the $a = -2$ case. After making these simplifications we are able to achieve the third point in our strategy. It appears that only for the $a = 4$ case can we solve our consistency conditions and can only obtain one nontrivial solution. Our systems of equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} U \\ W \end{pmatrix} = P_2 * \mathrm{grad}((D_1D_2U)U + \tfrac{4}{3}U^3 + (D_1D_2W)W - 2W^2U) \tag{54}$$

where

$$P_2 = \begin{pmatrix} D_1D_2\partial + s(U) & s(W) \\ s(W) & D_1D_2\partial + s(U) \end{pmatrix} \tag{55}$$

and $s(U)$, $s(W)$ are defined by (30).

Explicitly we obtain

$$U_t = \partial[-U_{xx} + 3(D_1U)(D_2U) + 6(D_1D_2U)U + 4U^3 + 3(D_2W)(D_1W) - 6W^2U] \tag{56}$$

$$W_t = \partial[-W_{xx} - 2W^3 + 3(D_2W)(D_1U) - 3(D_1W)(D_2U)] \\ -6(D_2W)(D_2U)U - 6(D_1W)(D_1U)U. \tag{57}$$

In the bosonic sector we obtain

$$u_{0t} = \partial[-u_{0xx} + 6u_1u_0 + 4u_0^3 - 6w_0u_0] \tag{58}$$

$$w_{0t} = \partial[-w_{0xx} - 2w_0^3] \tag{59}$$

$$u_{1t} = \partial[-u_{1xx} + 3u_1^2 + 3w_1^2 + 3w_{0x}^2 - 3u_{0x}^2 - 6u_{0xx}u_0 + 12u_1u_0^2 - 6w_0^2u_1 - 12w_1w_0u_0] \tag{60}$$

$$w_{1t} = \partial[-w_{1xx} + 6w_{0x}u_{0x} + 6w_1u_1 - 6w_1w_0^2] + 12w_1u_0u_{0x} - 12w_{0x}u_1u_0. \tag{61}$$

Interestingly, our Lax operator has a simple representation

$$L := [(D_1D_2 + U + W)(D_1D_2 + U - W)]^2. \tag{62}$$

This form of Lax operator suggests that we consider a much simpler Lax pair. Namely, it is sufficient to investigate the root of this Lax operator

$$L = (D_1D_2 + U + W)(D_1D_2 + U - W) \tag{63}$$

with the corresponding Lax pair

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -4\mathrm{i}[L, (L^{\frac{3}{2}})_+]. \tag{64}$$

If we further reduce the bosonic limit of our system of equations, by demanding that $u_0 = 0$ and $w_0 = 0$, we obtain the following system

$$u_{1t} = \partial(-u_{1xx} + 3u_1^2 + 3w_1^2) \tag{65}$$

$$w_{1t} = \partial(-w_{1xx} + 6u_1w_1) \tag{66}$$

which does not coincide with the Hirota–Satsuma equations (21), (22). Moreover we can transform these equations, to the system of two noninteracting Korteweg–de Vries equations using

$$u_1 \mapsto u_1 + w_1 \tag{67}$$

$$w_1 \mapsto u_1 - w_1. \tag{68}$$

However, we cannot do the same with the supersymmetric level.

It is rather an unexpected result, because our supersymmetrization method involves supersymmetrizations of the Hirota–Satsuma Lax operator. The observation that, in the process of the supersymmetrization, in the bosonic limit, we do not obtain the desired equation, is known in the theory of supersymmetrization of soliton's equation. It happens for example in the SUSY Boussinesq equation.

Finally, let us discuss the problem of the existence of integrals of motion in our model. We succeeded first in constructing three conservation laws, which are

$$I_1 = \int U \,\mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \tag{69}$$

$$I_3 = \int ((D_1 D_2 U)U + \tfrac{4}{3}U^3 + (D_1 D_2 W)W - 2W^2 U) \,\mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \tag{70}$$

$$\begin{aligned} I_5 = \int (16U^5 &+ 40(D_1 D_2 U)U^3 + 10(D_1 D_2 U)^2 U + 30U_x^2 U - 5(D_1 D_2 U_{xx})U \\ &-10(D_1 D_2 W)W^3 - 5(D_1 D_2 W_{xx})W + 20(D_2 W_x)(D_2 W)U \\ &+20(D_1 W_x)(D_1 W)U + 50(D_2 W)(D_1 W)U^2 + 20W_{xx}WU \\ &+30(D_1 D_2 W)^2 U - 30(D_1 D_2 W)WU^2 + 20W_x^2 U \\ &+30W^4 U - 3(D_1 D_2 U)W^2 U - 40W^2 U^3) \,\mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2. \end{aligned} \tag{71}$$

Moreover, we proved the absence of second, fourth and sixth conformal dimensional integrals of motion in our system. This is an unexpected result. It means that our equations of motion do not coincide, in the limit when $W = 0$, with the SUSY version of the KdV equation. Indeed, as we saw, the SUSY $a = 4$ KdV equation possesses odd and even conformal dimensional integrals of motion. It is interesting to note that in these two equations we have two different mechanisms of construction of the conserved currents. The pure SUSY $a = 4$ KdV equation has two nonequivalent roots which are responsible for the integrals of motion (see formulae (51), (52)). In the SUSY Hirota–Satsuma case we can only construct one root of the Lax operator and hence we have even dimensional currents.

5. Concluding remarks

We have constructed the supersymmetric analogue of the Lax operator responsible for the Hirota–Satsuma equation. Interestingly, this operator does not produce the classical Hirota–Satsuma equation. Moreover, we saw that it is possible to construct a SUSY system (38), (39) which contains a Hirota–Satsuma system without any references to the Lax operator. However, we did not investigate the integrability of this system, because we could not succeed in constructing the Lax operator, and we did not use any other criteria of the integrability (for example the Painlevé test). The classical systems which have Lax formulation are integrable and this is a benefit of the Lax approach. In the search of the SUSY Lax operator, for our system, we made the strong assumption that it reduces to the usual SUSY KdV Lax operator. We can drop this assumption, and assume that our

Lax operator produces the (SUSY) equation which is reduced in the bosonic limit to the Hirota–Satsuma equation. However, in this case, dificulties with the second Hamiltonian formulation of such a system appear. On the other hand the first Hamiltonian structure could appear and it needs further investigation. Note that $N = 2$ SUSY Virasoro algebra is uniquely defined and it strongly relies on the assumption on the structure of the Poisson tensor (3). In the classical Hirota–Satsuma case, the Poisson tensor (19) is reduced to the classical Poisson tensor connected with the Virasoro algebra. Hence, in the supersymmetric case, this assumption is reasonable and therefore our Lax operator should reduce to the KdV-type Lax operator. For such an operator, we investigated all possibilities in the class of two superbosonic fields.

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